

## Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <a href="http://about.jstor.org/participate-jstor/individuals/early-journal-content">http://about.jstor.org/participate-jstor/individuals/early-journal-content</a>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

#### On the Definition of Reducible Hypercomplex Number Systems.

II.

#### By Heman Burr Leonard.

§1.—Preliminary.—The present paper is intended as a completion of the problem studied in a former paper bearing the same title.\*

A hypercomplex number system is said to be reducible when, by a proper choice of units, it can be brought to the form

$$E \equiv E_i E_k \equiv e_1 \ldots e_m e_{m+1} \ldots e_n,$$

where the following conditions are fulfilled:

A), associativity of E;

 $C_1$ ),  $E_i$  forms a system by itself;

 $C_2$ ),  $E_k$  forms a system by itself;

$$(C_{jk}), e_j e_k = 0, \qquad j = 1, \ldots, m;$$

$$C_{jk}$$
),  $e_j e_k = 0$ ,  $j = 1, \ldots, m$ ;  $C_{kj}$ ),  $e_k e_j = 0$ ,  $k = m + 1, \ldots, n$ .

In the former paper were listed seventy-eight different definitions from which the above requirements can be deduced, and these definitions were based on the following twenty conditions:

- $(J_1 \ J_2) \ J_3 = J_1 \ (J_2 \ J_3);$  $A_1$ ),  $(K_1K_2)K_3 = K_1(K_2K_3);$  $A_2$ ),  $(K_1 J_1) J_2 = K_1 (J_1 J_2);$  $A_3$ ),  $(J_1 K_1) K_2 = J_1 (K_1 K_2);$  $A_4$ ),  $(J_1 K_1) J_2 = J_1 (K_1 J_2);$  $A_5$ ),  $(K_1J_1) K_2 = K_1 (J_1 K_2);$  $A_6$ ),  $(J_1 J_2) K_1 = J_1 (J_2 K_1);$  $A_7),$  $(K_1 K_2) J_1 = K_1 (K_2 J_1);$  $A_8$ ),  $C_1$ ),  $E_j$  is closed under multiplication, that is,  $J_1 J_2 = J_3$ ;
- $C_2$ ),  $E_k$  is closed under multiplication, that is,  $K_1 K_2 = K_3$ ;

<sup>\*</sup> Epsteen-Leonard, On the Definition of Reducible Hypercomplex Number Systems, American Mathematics, Vol. XXVII (1905), p. 217.

$$egin{array}{lll} C^{j}_{jk}, & J_{1} \ K_{1} = K_{2} & (J_{2} = 0); \ C^{k}_{jk}, & J_{1} \ K_{1} = J_{2} & (K_{2} = 0); \ C^{k}_{kj}, & K_{1} \ J_{1} = K_{2} & (J_{2} = 0); \ K_{1} \ J_{1} = J_{2} & (K_{2} = 0); \ \end{array}$$

 $C_r$ ), right-hand division possible and unique, that is, not every X is a right-hand divisor of zero; hence, an X exists such that

$$X_1 X = 0$$
, only if  $X_1 = 0$ ;

 $C_i$ ), left-hand division possible and unique, that is, not every X is a left-hand divisor of zero; hence, an X exists such that

$$XX_1 = 0$$
, only if  $X_1 = 0$ ;

 $(C_r^j)$ , right-hand division possible and unique in the subset  $E_j$ , that is, not every J is a right-hand divisor of zero; hence, a J exists such that

$$J_1 J = 0$$
, only if  $J_1 = 0$ ;

 $C_i^j$ ), left-hand division possible and unique in the subset  $E_j$ , that is, not every J is a left-hand divisor of zero; hence, a J exists such that

$$JJ_1 = 0$$
, only if  $J_1 = 0$ ;

 $C_r^k$ ), right-hand division possible and unique in the subset  $E_k$ , that is, not every K is a right-hand divisor of zero; hence, a K exists such that

$$K_1 K = 0$$
, only if  $K_1 = 0$ ;

 $C_l^k$ ), left-hand division possible and unique in the subset  $E_k$ , that is, not every K is a left-hand divisor of zero; hence, a K exists such that

$$KK_1 = 0$$
, only if  $K_1 = 0$ .

Conditions  $A_1$ ,  $A_2$ ,  $A_3$ ,  $A_4$ ,  $A_5$ ,  $A_6$ ,  $A_7$ , and  $A_8$ , together are equivalent to the associativity condition

$$A$$
),  $(X_1 X_2) X_3 = X_1 (X_2 X_3)$ ,

where  $X_p$  (p = 1, 2, 3) are any numbers of the system E and where

$$X_p = J_p + K_p = \sum_{j_1=1}^m x_{pj_1} e_{j_1} + \sum_{k_1=m+1}^n x_{pk_1} e_{k_1}.$$

It was shown also that the conditions composing each definition are independent.

By systematic checking I have found that the ninety-six definitions given in the following tables, together with the seventy-eight of the former paper, give all possible methods of defining reducibility that depend upon the above twenty assumptions. In this paper I use throughout the notation and the tables of the former paper; the definitions being based of course upon Table I. Table II. being complete, no additions are made; Tables III<sub>2</sub>. and  $IV_2$ . of this paper supplement Tables III. and IV. of the former paper. In the notation  $R_{13_1}$ , the subscript indicates that this is the first definition which is to be inserted after  $R_{13}$  of the former paper to form the sequence as now completed. Definition  $R_{13_2}$  immediately follows  $R_{13_1}$ .

Independence Proofs.—With the exception of those indicated by a (\*) in the tables, the conditions composing the various definitions are independent. The independence proofs are easily verified by Table V., which is reprinted from the former paper.

§2.—Definitions of Reducibility by Independent Assumptions. We reproduce Table I. from the former paper.

We re	produce Table I. from the former paper.	
	TABLE I.	
Notation.	Assumptions.	Consequenc

Notation.		Assumptions.							•
$D_1$	$oxed{A_7,\ C^j_{jk},\ C^k_{jk},}$	$C_{kj}^i,\ C_{kj}^k,$	$C_r$						$C_{1}$
$D_2$	$igg _{A_3,\;C^j_{jk},\;C^k_{jk}}$	, $C^j_{kj}$ , $C^k_{kj}$ ,	•	$C_{l}$					$C_{1}$
$D_{\mathtt{3}}$	$igg  A_7, \ C^j_{jk}, \ C^k_{jk},$			$C_{\bullet}^{\scriptscriptstyle A}$	•				$C_{1}$
$D_4$	$A_3$ ,	$C^j_{kj},\ C^k_{kj},$			$C_l^{k}$				$C_{\mathfrak{1}}$
$D_{\mathfrak{b}}$	$A_8, C^j_{jk}, C^k_{jk},$	$C^j_{kj},\ C^k_{kj}$ ,	$C_r$						$C_{2}$
$D_{6}$	$A_4, C_{jk}^j, C_{jk}^k,$	$C^j_{kj},\ C^k_{kj},$		$C_l$					$C_{2}$
$D_{7}$	$A_8$ ,	$C^j_{kj},\ C^k_{kj}$		$C_r^j$					$C_{2}$
	$A_4$ , $C^j_{jk}$ , $C^k_{jk}$ ,			$C_l^i$					$C_{\mathtt{2}}$
$D_{\mathfrak{d}}$	$A_5$ , $A_6$ ,	$C^i_{kj},\ C^k_{kj},$		$C_{r}^{j}$		$C^{j}_{m{j}m{k}}$			
$D_{10}$	$A_6$ ,	$C^j_{kj},\ C^k_{kj},$			$C_l^k$	O	jk jk		
$D_{11}$	$egin{aligned} A_6, \ C^{i}_{jk}, \ C^{k}_{jk}, \ C^{k}_$			$C'_{\bullet}$	k r		$C_{kj}^k$		
$D_{12}$	$A_5, C^j_{jk}, C^k_{jk},$			$C_l^j$				$C^j_{kj}$	,

I.—From a consideration of this table, it is evident that  $D_6$ ,  $D_5$ ,  $D_2$ ,  $D_1$  are the only dependencies in which the division assumptions are on the system

E as a whole. There are eight possible combinations of these that give a definition of reducibility and they all appear in Table II.

II.—In  $D_{12}$ ,  $D_{11}$ ,  $D_{10}$ ,  $D_{9}$ ,  $D_{8}$ ,  $D_{7}$ ,  $D_{4}$ , and  $D_{3}$  the division assumptions are on the subsets  $E_{j}$  and  $E_{k}$ . The thirty-eight definitions of *Table III*. with the additional twenty-four of *Table III*<sub>2</sub>. give their sixty-two possible combinations.

TABLE III2.

(1)	(2)			(8)			Proved b	oy (3)	(6)
Nota- tion.	From Table I.		$\mathbf{A}\mathbf{s}$	sumptions.			(4)	(5)	Proved by (3) and (4).
	$D_{12}$	$A_5$	$egin{array}{ccc} C^{j}_{jk} & C^{k}_{jk} \ C^{j}_{jk} & C^{k}_{jk} \end{array}$		$C_l^i$		$C^{j}_{kj}$		
70	$D_{11}$	$A_6$	$C^{j}_{jk} \ C^{k}_{jk}$		$\epsilon$	yk r	$C^{k}_{kj}$		
$R_{13_1}$	$D_7$	$A_8$			$C_{r}^{j}$		"		$C_2$
		$C_1$	$egin{array}{ccc} C^i_{jk} & C^k_{jk} \ C^j_{jk} & C^k_{jk} \end{array}$						
	$D_{12}$	$A_5$	$C^{i}_{jk} \ C^{k}_{jk}$		$C_i^{i}$		$C^{j}_{k\!j}$		
$R_{ m 13_o}$	$ D_{11} $	$A_6$	$C^j_{jk}  C^k_{jk}$		6	$r^{k}$	$C^k_{\it kj}$	,	
		(	$\mathcal{O}_{2}$						
	$\mid D_{\iota} \mid$	$A_2$				$C_l^k$	66 66		$C_1$
	$oxed{D_{12}}$	$A_5$	$C^j_{jk} \ C^k_{jk}$		$C_l^{m j}$		$C^j_{kj}$		
$R_{\scriptscriptstyle 15.}$				$C^k_{kj}$					
101	$D_8$	$A_4$	$C^j_{jk}  C^k_{jk}$		$oldsymbol{C}_{l}^{i}$			$C_2$	
	$D_4$	$A_3$		$C^k_{\it kj}$		$C_l^k$	4.6		$C_1$
	$D_{12}$	$A_5$	$C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$		$oldsymbol{C}_l^j$		$C^{j}_{kj}$		
$R_{17}$	$D_7$			$C^k_{kj}$					
11]	$D_7$	$A_8$		$C^{k}_{kj}$	$C^j_r$		"		$C_2$
	$D_4$	$A_3$		$C^k_{kj}$		$C_l^k$	"		$C_1$
	$D_{12}$	$A_5$	$C^j_{jk}  C^k_{jk}$		$C_l^j$		$C^{j}_{kj}$		
$R_{17}$				$C^k_{kj}$					
112	$D_7$ $D_3$	$A_8$		$C^k_{k\!j}$	$C^j_{r}$		"		$C_2$
	$\mid D_3 \mid$	$A_7$	$C^j_{jk}$ $C^k_{jk}$		(	$\mathcal{I}_{\mathbf{r}}^{k}$		$C_1$	

TABLE III<sub>2</sub>.—Continued.

(1)	(2)			(3)			Proved	by (3)	(6)
Nota- tion.	From Table I.		Assı	imptions.			(4)	(5)	Proved by (8) and (4).
$R_{17_3}$	$egin{array}{c} D_{12} \ \dots \ D_{7} \ \dots \end{array}$	$egin{array}{cccc} A_5 & & & & & & & & & & & & & & & & & & &$	$C^j_{jk}  C^k_{jk}$	$egin{array}{c} C_{kj}^k \ C_{kj}^k \end{array}$	$C_r^j$	i I	$C^{j}_{k j}$		$C_2$
$R_{17_4}$	$D_{12}$ $\dots$ $D_{n}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C^j_{jk}$ $C^k_{jk}$	$egin{aligned} C_{kj}^k \ C_{kj}^k \end{aligned}$	c	$C_{l}^{k}$	$C^{i}_{kj}$		$C_1$
$R_{19_1}$	$egin{array}{c} D_{11} \ D_8 \ D. \end{array}$	$egin{array}{c} A_6 \ A_4 \ A \end{array}$	$C^j_{jk}$ $C^k_{jk}$ $C^j_{jk}$ $C^k_{jk}$	O kij Vici	Ó	$C^k_{m{r}}$ $C^k_{m{r}}$	$C^k_{kj}$	$C_2$	$C_1$
$R_{21_1}$	$D_4$ $D_{11}$ $D_7$ $D_4$	$egin{array}{c} A_6 \ A_8 \ A_3 \end{array}$	$C^j_{jk}  C^k_{jk}$	ri kj Ti kj Vi kj	$C_r^j$	$C^k_{m{r}}$ $C^k_{m{t}}$	$egin{aligned} C_{kj}^k \ & `` \ & `` \end{aligned}$		$C_{1}$
$R_{ m 2l_2}$	$\mid D_7 \mid$	$egin{array}{c} A_6 \ A_8 \ A_7 \end{array}$	$C^{j}_{jk}$ $C^{k}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$	Cij Kij Nj Kij	$C^j_{\ m{r}}$	$egin{array}{c} C_{r}^{k} \ C_{r}^{k} \end{array}$	$C^k_{kj}$ , ,	$C_1$	$C_2$
$R_{2\mathbf{l}_3}$	$D_{11}$ $D_{7}$	A	$\Omega i \Omega k$	<sup>A</sup> ks	$C^j_r$	$C_r^k$	$C^k_{kj}$		$C_2$
$R_{21_4}$	$D_{11}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C^{j}_{jk}  C^{k}_{jk}$	is is		$C_r^k$	$C_{kj}^k$		$C_1$
	$D_4$	$A_3$	(	$\mathcal{I}_{kj}^{\ j}$		$C_l^k$	"		$C_1$

TABLE III<sub>2</sub>.—Continued.

(1)	(2)			(3)			Proved	by (3)	(6)
Nota- tion.	From Table I.			Assumptions.			(4)	(5)	Proved by (3) and (4).
	$D_{10}$	$A_6$		$C^j_{kj} \ C^k_{kj}$		$C_{l}^{k}$	$C^k_{jk}$		
$R_{or}$	$D_9$	$A_5$		$C^i_{kj} \ C^k_{kj}$	$C_r^j \ C_l^j$		$C^{j}_{jk}$		
231	$D_8$	$egin{array}{c} A_5 \ A_4 \ C_1 \end{array}$					"		$C_2$
	$D_{10}$	$A_6$		$egin{array}{ccc} C_{kj}^j & C_{kj}^k \ C_{kj}^j & C_{kj}^k \end{array}$		$C_l^k$			
$R_{\infty}$	$D_{\mathfrak{g}}$	$A_5$	0.	$C^{j}_{kj}  C^{k}_{kj}$	$C_{m{r}}^{\jmath}$		$C^j_{jk}$		
291	$D_3$	$A_{7}$	$C_2$	$egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ C_{kj}^j & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k & C_{kj}^k \ \end{array} \ egin{array}{lll} C_{kj}^i & C_{kj}^k & C_{kj}^k & C_{kj}^k & C_{kj}^k & C_{kj}^k \ \end{array}$		$C_r^k$	"		$C_1$
	$D_{10}$	$A_6$	C	$C^j_{kj}  C^k_{kj}$		$C_l^k$	$C^k_{jk}$		
$R_{ m 30_1}$	D	4	$C_{jk}$		$C_{l}^{j}$				$C_2$
	$egin{array}{c} D_8 \ D_4 \ \end{array}$	$A_3$	$\mathcal{O}_{jk}$	$C^j_{kj}  C^k_{kj}$	$O_{l}$	$C_i^k$		$C_1$	
	$D_{10}$	$A_6$		$C^j_{kj}  C^k_{kj}$		$C_l^k$	$C^k_{jk}$		
$R_{20}$			$C^{j}_{jk}$						
302	$D_8$	$A_4$	$C^j_{jk}$		$C_l^j$	$C^k_r$	66		$C_2$
	$D_3$	$A_7$	$C^{j}_{jk}$			$C_r^k$	"		$C_1$
	$D_{10}$	$A_6$		$C^j_{kj}  C^k_{kj}$		$C_l^k$	$C^k_{jk}$		
$R_{ m 30_3}$			$C^{j}_{jk}$						
	$D_8$	$A_4$	$C^{j}_{jk}$		$C_l^j$		"		$C_2$
	_	. 01		out of					
	$ig egin{array}{c c} D_{10} \ \end{array}$	$A_6$	~*	$C_{kj}^{g}$ $C_{kj}^{\kappa}$		$C_l^k$	$C^k_{jk}$		
$R_{ m 31_1}$	$egin{array}{c} D_7 \ D_3 \ \end{array}$		$C^{j}_{jk}$	$C^{i}_{kj}  C^{k}_{kj}$	o:				
	$D_7$	$A_8$	د	$C_{kj}^{\eta} \ C_{kj}^{\kappa}$	$C_{r}^{g}$	~1		$C_2$	
-	$\mid D_3 \mid$	$A_7$	$C^{j}_{jk}$			$C_r^k$		<u> </u>	$C_1$

TABLE III<sub>2</sub>.—Continued.

(1)	(2)		(3)		Proved	by (3)	(6)
Nota- tion.	From Table I.	Assu	mptions.		(4)	(5)	Proved by (3) and (4).
$R_{33_1}$	$D_{10}$ $\dots$		$C_{kj}^k$	$C_l^k$	$C^k_{jk}$		
10331	$D_3$	$egin{array}{ccc} & C^j_{jk} & & & & & & & & & & & & & & & & & & &$		$C_r^k$	"		$C_{1}$
$R_{\scriptscriptstyle 34.}$	$D_9$	$egin{aligned} C^k_{jk} \ A_5 \end{aligned} \qquad \qquad C^g_{k} \end{aligned}$	$_{j}$ $C_{kj}^{k}$ $C_{r}^{j}$ $C_{r}^{l}$		$C^{j}_{jk}$		
041	$egin{array}{c c} D_8 \ D_4 \ \end{array}$	$egin{array}{cccc} oldsymbol{A_4} & & C^k_{jk} \ oldsymbol{A_3} & & C^j_k \end{array}$	$_{j}$ $C_{kj}^{k}$	$C_l^k$	<b>( 6</b>	$C_1$	$C_2$
T)	$egin{array}{c} \dots \ D_{9} \end{array}$	$C^k_{jk} \ A_{f 5} \ C^j_k$	$_{j}$ $C_{kj}^{k}$ $C_{r}^{j}$ $C_{l}^{j}$		$C^j_{jk}$		
$R_{34_2}$	$egin{array}{c c} D_8 \ D_3 \end{array}$	$egin{array}{lll} A_4 & & C^k_{jk} \ A_7 & & C^k_{jk} \end{array}$	$C_l^{i}$	$C_r^k$	"		$egin{array}{ccc} C_{2} & & & & & & & & & & & & & & & & & & $
$R_{34_3}$	$egin{array}{c} \dots \ D_{9} \ D_{8} \end{array}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$_{j}$ $C_{kj}^{k}$ $C_{r}^{j}$ $C_{r}^{l}$		$C^{b}_{jk}$		$C_2$
	 D	$C_1$ $C_{jk}^k$ $C_{jk}^k$			$C^{j}_{jk}$		
$R_{ exttt{35}_1}$	$D_9$ $D_7$ $D_3$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$_{j}$ $C_{kj}^{k}$ $C_{r}^{j}$	$C_r^k$	<i>,</i> ,	$C_2$	$C_1$
$R_{ m 37_1}$	$egin{bmatrix} \dots \ D_{9} \end{bmatrix}$	$egin{aligned} A_5 & & & C_{jk}^k \ & & & C_2 \end{aligned}$	$_{j}$ $C_{kj}^{k}$ $C_{r}^{j}$		$C^{j}_{jk}$		
	$D_3$	$A_7$ $C_{jk}^k$		$C_r^k$	"		$C_1$

III.—The thirty-two definitions of  $Table\ IV$ , and the additional seventy-two of  $Table\ IV_2$ , each contain at least one division assumption upon the system E as a whole and at least one division assumption on one of its subsets.

TABLE  $IV_2$ .

(1)	(2)					(3)				P	roved by	7 (3)	(6)
Nota- tion.	From Table I.				Assı	umption	18.			(	4)	(5)	Proved by (8) and (4).
	$D_{12}$	$A_{5}$		$C^{j}_{jk}$	$C^k_{jk}$			$C_l^j$	(	$\mathcal{D}_{kj}^{j}$			
$R_{\scriptscriptstyle{F}}$	$egin{array}{c} D_{12} \ D_{11} \ D_6 \ \end{array}$	$A_6$		$C^j_{jk}$	$C^k_{jk}$			$C_r^k$			$C_{kj}^k$		
541	$D_6$	$A_4$		$C^j_{jk}$	$C^k_{jk}$		$C_{l}$				"		$C_2$
	,												
	$egin{array}{c} D_{12} \ D_{11} \ D_{5} \ \end{array}$	$A_5$		$C^j_{jk}$	$C^k_{jk}$			$C_l^j$	0	rj kj			
$R_{\mathfrak{58}_1}$	$D_{11}$	$A_6$		$C^j_{jk}$	$C^k_{jk}$			$C^k_{r}$			$C_{kj}^k$		
	$D_5$	$A_8$		$C^j_{jk}$	$C^k_{jk}$	(	$\mathcal{O}_{r}$			"	"		$C_2$
		(	$C_{1}$										
	$D_{12}$	$A_{5}$	$C_2$	$C^j_{jk}$	$C^k_{jk}$			$C_i^j$	0	) kj			
D	$D_{11}$	$A_6$		$C^j_{jk}$	$C^{m k}_{jm k}$			$C_r^k$			$C_{kj}^k$		
$n_{58_2}$			$C_{2}$										
	$\mid D_2 \mid$	$A_3$		$C^j_{jk}$	$C^k_{jk}$		$C_{i}$			4	"		$C_1$
	$D_2$ $D_{12}$ $D_{11}$	$A_{5}$		$C^{j}_{jk}$	$C^k_{jk}$			$C_l^j$	0	$\mathcal{I}_{kj}^{j}$			
$R_{\scriptscriptstyle 58_3}$	$D_{11}$	$A_6$		$C^j_{jk}$	$C^k_{jk}$			$C_r^k$			$C_{kj}^k$		
•			$C_2$										
	$D_1$	$A_7$		$C_{jk}^{j}$	${C}^{\kappa}_{jk}$	(	$C_{r}$		-   '	: 4	"		$C_1$
	$D_{12}$	$A_5$		$C^{j}_{jk}$	$C^k_{jk}$			$C_{\it i}^{\it j}$	1	)j kj			
p						$C_{kj}^k$							
$R_{\mathfrak{58}_{4}}$	$D_8$	$A_4$			$C^k_{jk}$			$C_l^{ij}$				$C_{2}$	
	$D_2$	$A_3$		$C^j_{jk}$	$C^k_{jk}$	$C^k_{k\!j}$	$C_{l}$		'	6			$\int C_1$

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)			(3)				Proved b	oy (3)	(6)	)
Nota- tion.	From Table I.		Assur	nptions.				(4)	(5)	Prov by (3) and	ed (4).
	$D_{12}$		$C^{j}_{jk}$ $C^{k}_{oldsymbol{j}k}$		***************************************	$C_l^j$		$C^{j}_{kj}$			
${m R}$				$C^{k}_{kj}$							
±€58 <sub>5</sub>	$D_8$	$A_4$	$C^j_{jk}  C^k_{jk}$			$C_l^j$			$C_2$		
	$D_1$	$A_7$	$egin{array}{ccc} C^j_{jk} & C^k_{jk} \ C^g_{jk} & C^k_{jk} \end{array}$	$C_{kj}^{k}$ $C_{r}$	·			"		$C_1$	
	$D_{12}$	$A_5$	$C^j_{jk}  C^k_{jk}$			$C_l^j$		$C^{j}_{kj}$			
D				$C^k_{k\!j}$							
I 1 586	$D_{7}$	$A_8$		$C^k_{kj}$	$C_{m{r}}^{j}$			"			$C_{2}$
	$D_2$	$A_3$	$C^{j}_{jk}  C^{k}_{jk}$	$C_{kj}^k$	$C_{m{i}}$			66		$C_1$	
	$D_{12}$	$A_{5}$	$C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$			$C_{m{l}}^{j}$		$C^j_{kj}$			
D				$C^k_{k\!j}$							
$n_{58_7}$	$D_7$	$A_8$		$C^k_{k\!j}$	$*C_r^j$			44			$C_{2}$
	$D_1$	$A_7$	$C^j_{jk}$ $C^k_{jk}$	$C_{kj}^k$ $C_r$				"		$C_1$	
	$D_{12}$	$A_5$	$C^{i}_{jk}$ $C^{k}_{jk}$ $C^{i}_{jk}$ $C^{k}_{jk}$ $C^{i}_{jk}$ $C^{k}_{jk}$			$C_{\it l}^{j}$		$C^j_{kj}$			
מ				$C^{k}_{kj}$							
$R_{58_8}$	$D_6$	$A_4$	$C^j_{jk}  C^{k}_{jk}$	$C_{kj}^{m k}$	$C_{l}$			"			$C_2$
	$D_4$	$A_3$		$C_{kj}^k$			$*C_l^k$	"		$C_{1}$	
	$D_{12}$	$A_5$	$C^j_{jk}  C^k_{jk}$			$C_l^j$		$C^j_{kj}$			
D				$C_{kj}^k$							
$K_{58_9}$	$D_6$	$A_4$	$C^j_{jk}  C^k_{jk}$	$C_{kj}^{k}$	$C_{i}$			"			$C_{2}$
	$D_3$		$C^{j}_{jk}  C^{k}_{jk}$	-		C	k r		$C_1$		

TABLE  $IV_2$ .—Continued.

(1)	(2)			(3)			Proved	by (8)	(6)	
Nota- tion.	From Table I.		Assu	mption	18.		(4)	(5)	Prove by (3) and	(4).
	$D_{12}$	$A_5$	$C^{j}_{jk}$ $C^{k}_{jk}$			$C_l^j$	$C^j_{kj}$			
$R_{\scriptscriptstyle \sf FO}$				$C^k_{k\!j}$						
9010	$D_6$	$A_4$	$C^j_{jk}  C^k_{jk}$	$C_{kj}^{k}$	$\mathit{C}_{i}$		"			$C_2$
	$D_2$	$A_3$	$C^j_{jk}  C^k_{jk}$	$C_{kj}^{k}$	$\mathit{C}_{\!i}$		"		$C_1$	
	$D_{12}$	$A_5$	$C^j_{jk}  C^k_{jk}$			$C_{m{i}}^{j}$	$C^{j}_{kj}$			
$R_{\kappa \circ}$				$C^{k}_{kj}$						
9011	$D_6$	$A_{4.}$	$C^j_{jk} \ C^k_{jk}$	$C_{kj}^{k}$	$C_{l}$		"			$C_{2}$
	$D_1$	$A_7$	$C^j_{jk} \ C^k_{jk}$	$C_{\it kj}^{\it k}$	$C_{r}$		"		$C_1$	
	$D_{12}$	$A_5$	$C^{j}_{jk} \ C^{k}_{jk}$			$C_l^j$	$C_{kj}^{j}$			
$R_{\scriptscriptstyle  extsf{Fo}}$				$C^k_{k\! j}$						
0012	$D_6$	$A_4$	$C^j_{jk} \ C^k_{jk}$	$C_{kj}^{k}$	$C_{l}$		"			$C_{2}$
		$C_1$								
	$D_{12}$	$A_5$	$C^j_{jk}  C^k_{jk}$			$C_l^j$	$C^{j}_{kj}$			
n				$C_{kj}^k$						
$H_{58_{18}}$	$D_{5}$	$A_8$	$C^j_{jk} \ C^k_{jk}$	$C_{kj}^k$	$C_r$		66			$C_{2}$
	$D_4$	$A_3$		$C_{kj}^k$			$C_l^k$ "		$C_{\mathbf{i}}$	
	$D_{12}$	$A_5$	$C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$			$C_l^{j}$	$C_{kj}^{j}$			
D				$C_{k\!j}^k$						
IV 5814	$D_5$	$A_8$	$C^j_{jk} \ C^k_{jk}$	$C^{m{k}}_{m{k}m{j}}$	$C_{r}$		"			$C_{2}$
	$D_{3}$	$A_7$	$C^j_{jk}$ $C^k_{jk}$			$*C_r^k$		$C_1$		

TABLE  $IV_2$ .—Continued.

(1)	(2)			(3)		Proved	by (3)	(6)
Nota- tion.	From Table I.		Assu	imptions.		(4)	(5)	$ \begin{array}{c} \text{Proved} \\ \text{by} \\ (3) \text{ and } (4). \end{array} $
	$D_{12}$	$A_5$	$C^j_{jk} \ C^k_{jk}$		$C_l^{i}$	$C^{j}_{kj}$		
$R_{ exttt{58}_{ exttt{15}}}$	$D_{\scriptscriptstyle{5}}$	$A_{\mathrm{s}}$	$C^j_{ik}  C^k_{ik}$	$egin{array}{c} C_{kj}^{\ \kappa} \ C_{ki} \end{array}$		"		$C_2$
	$oxedsymbol{D_2}$	$A_3$	$C^j_{jk}  C^k_{jk}$	$C_{kj}^{k}$ $C_{l}$		64		$igcap C_1$
	$D_{12}$	$A_5$	$C^{j}_{jk}  C^{k}_{jk}$		$C_{m{i}}^{m{j}}$	$C^{j}_{k j}$		
D				$C_{kj}^{k}$				
1 v 58 <sub>16</sub>	$oxedsymbol{D_5}$	$A_8$	$C^j_{jk}  C^k_{jk}$	$C^k_{kj}$ $C_r$		"		$C_2$
	$oxedsymbol{D_1}$	$A_7$	$C^{j}_{jk}  C^{k}_{jk}$	$C_{kj}^k C_r$		"		$C_1$
	$D_{12}$	$A_5$	$C^{j}_{jk}  C^{k}_{jk}$		$C_l^j$	$C^{j}_{kj}$		
$\boldsymbol{\mathcal{D}}$				$C_{kj}^{k}$				
£658 <sub>17</sub>	$oxed{D_5}$	$A_8 = C_1$	$C^{j}_{jk}$ $C^{k}_{jk}$	$C_{kj}^{\ k} \ C_r$				$C_{2}$
	$D_{10}$	A.	$C^{j}_{jk}  C^{\kappa}_{jk}$		$C_{t}^{g}$	$C^{j}_{kj}$		
			$\circ jk \circ jk$	$C_{ki}^{\;k}$	o t	- KJ		
$R_{\mathfrak{58}_{18}}$			$C_{2}$	70)				
	$D_2$	$A_3$	$C^{j}_{jk}  C^{k}_{jk}$	$C_{kj}^k  C_l$		66		$C_1$
	$D_{12}$	$A_5$	$C^j_{jk}  C^k_{jk}$		$C_l^j$	$igg  C^j_{kj}$		
$R_{ m 58_{19}}$			a	$C_{kj}^{k}$				
5019	$D_1$	$A_7$	$C_{m{z}}$ $C^{j}_{jk}$ $C^{k}_{jk}$	$C_{kj}^k C_r$		"		$C_1$

TABLE IV2.—Continued.

(1)	(2)		(3)			Proved b	y (3)	(6)
Nota- tion.	From Table I.		Assumpt	ions.		(4)	(5)	Proved by (3) and (4).
			$C^{j}_{kj}$					
$R_{ro}$	$\mid D_{11} \mid$	$A_6$	$egin{array}{ll} C^{j}_{jk} & C^{k}_{jk} \ C^{j}_{jk} & C^{k}_{jk} \ C^{j}_{jk} & C^{k}_{jk} & C^{j}_{kj} \end{array}$		$C_{r}^{k}$	$C_{kj}^k$		
	$\mid D_8 \mid$	$A_4$	$C^j_{jk} \ C^k_{jk}$		$^*C_l^j$		$C_2$	
	$\mid D_2 \mid$	$A_3$	$C^j_{jk}$ $C^k_{jk}$ $C^j_{kj}$	$C_{\iota}$		66		$C_1$
			$C_{kj}^{j}$					
D	$D_{11}$	$A_6$	$C^j_{jk}  C^k_{jk}$		$C_r^k$	$C_{kj}^k$		
11 58 <sub>21</sub>	$D_8$	$A_4$ .	$C^j_{jk}  C^k_{jk}$		$C_l^{ij}$		$C_2$	
	$D_1$	$A_7$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_r$		66		$C_1$
			$C^{j}_{kj}$					
D	$D_{11}$	$A_6$	$C^j_{jk}  C^k_{jk}$		$C^k_{m{r}}$	$C_{kj}^k$		
$R_{58_{22}}$	$\mid D_7 \mid$	$A_8$	$C^j_{kj}$	(	$\mathcal{O}_{m{r}}^{j}$	"		$C_{2}$
	$oxedsymbol{D_2}$	$A_3$	$C^j_{jk}$ $C^k_{jk}$ $C^j_{kj}$	$C_{m{\imath}}$		"		$C_1$
			$C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{kj}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{kj}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{kj}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{kj}$					
-	$D_{11}$	$A_6$	$C^j_{jk}  C^{k}_{jk}$		$C_r^{k}$	$C_{kj}^k$		
$R_{ m 58_{23}}$	$\mid D_{7} \mid$	$A_8$	$C^{j}_{kj}$	*	$C^j_{m{r}}$	"		$C_2$
	$D_1$	$A_7$	$egin{array}{cccc} C^{j}_{jk} & C^{k}_{jk} & & & & & & & & & & & & & & & & & & &$	$C_{r}$		"		$C_1$
			$C^{i}_{jk} \;\; C^{k}_{jk} \;\; C^{i}_{kj} \;\; \ C^{j}_{jk} \;\; C^{j}_{jk} \;\; C^{j}_{jk} \;\; \ C^{j}_{jk} \;\; C^{j}_{kj} \;\; \ C^{j}_{kj} \;\; \ C^{j}_{kj} \;\; \ C^{j}_{kj} \;\; \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \$					
70	$D_{11}$	$A_6$	$C^j_{jk}  C^k_{jk}$		$C_r^k$	$C_{kj}^{\ k}$		
$R_{{f 58}_{24}}$	$D_6$	$A_4$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_l$		"		$C_2$
	$igg D_4$	$A_3$	$C_{kj}^{j}$		$^*C^k_l$	66		$C_1$

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)		(3)			Proved 1	by (3)	(6)
Nota- tion.	From Table I.		Assump			(4)	(5)	Proved by (3) and (4).
			$C_{kj}^{j}$					
ъ	$D_{11}$	$A_6$	$egin{array}{ll} C^{j}_{jk} & C^{k}_{jk} \ C^{j}_{jk} & C^{k}_{jk} & C^{j}_{kj} \ C^{j}_{jk} & C^{k}_{jk} \end{array}$		$C_r^k$	$C^k_{kj}$		
$H_{58_{25}}$	$D_6$	$A_4$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_l$		"		$C_2$
					$C^k_r$		$C_1$	
			$C^{j}_{kj} \ C^{j}_{jk} \ C^{k}_{jk} \ C^{j}_{jk} \ C^{j}_{jk} \ C^{j}_{kj} \ C^{$					
<b>T</b> D	$D_{11}$	$A_6$	$C^j_{jk} \ C^k_{jk}$		$C^k_{m{r}}$	$C^k_{kj}$		
$R_{58_{26}}$	$D_6$	$A_4$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_{l}$		"		$C_2$
	$D_2$	$A_3$	$C^j_{jk}$ $C^k_{ik}$ $C^j_{kj}$	$C_l$		"		$C_1$
			$egin{array}{ccc} C^j_{kj} \ C^j_{jk} & C^k_{jk} \ C^j_{jk} & C^k_{kj} \end{array}$					
	$D_{11}$	$A_6$	$C^j_{jk} \ C^k_{jk}$		$C_r^k$	$C_{kj}^{\ k}$		
$R_{\scriptscriptstyle 58_{27}}$	$D_{6}$	$A_4$	$C^j_{jk}$ $C^k_{jk}$ $C^j_{kj}$	$C_l$		"		$C_2$
	$D_1$	$A_7$	$C^j_{jk}  C^k_{jk}  C^j_{kj}$	$C_r$		"		$C_1$
			$C^{j}_{kj}$					
	$D_{11}$	$A_6$	$egin{array}{c} C^j_{kj} \ C^j_{jk} \ C^j_{jk} \ C^j_{jk} \ C^k_{kj} \end{array}$		$C_r^k$	$C_{kj}^{k}$		
$R_{\mathfrak{58}_{28}}$	$D_6$	$A_4$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_{i}$		"		$C_2$
		$C_{1}$						
			$C^j_{kj}$					
<b>3</b>	$D_{11}$	$A_6$	$C^j_{jk} \ C^k_{jk}$		$C_r^k$	$C_{kj}^{\;k}$		
$R_{\scriptscriptstyle 58_{29}}$	$D_5$	$A_8$	$egin{array}{ccc} C^j_{jk} & C^k_{jk} \ C^j_{ik} & C^j_{jk} & C^j_{kj} \ C^j_{kj} & C^j_{kj} \end{array}$	$C_r$		"		$C_2$
	$D_4$	$A_3$	$C_{kj}^{j}$		$C_{l}^{k}$	"		$C_1$

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)		(3)	ı		Proved	by (3)	(6)
Nota- tion.	From Table I.		Assump			(4)	(5)	Proved by (3) and (4).
-		•	$C^j_{kj}$			Name and the same		
D	$D_{11}$	$A_6$	$C^j_{jk}  C^k_{jk}$		$C_r^{k}$	$C_{kj}^{\ k}$		
IL 5830	$ig D_{\scriptscriptstyle 5}$	$A_8$	$C^{j}_{jk}$ $C^{j}_{jk}$ $C^{j}_{jk}$ $C^{j}_{kj}$	$C_r$		"		$C_2$
	$\mid D_3 \mid$	$A_7$	$C^{j}_{jk}$ $C^{\kappa}_{jk}$		$C_r^k$		$C_1$	
			$C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{jk}$ $C^{k}_{jk}$ $C^{j}_{kj}$					
R	$D_{11}$	$A_6$	$C^j_{jk}  C^k_{jk}$		$C_r^k$	$C_{kj}^k$		
<b>1</b> 058 <sub>31</sub>	$D_5$	$A_8$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_{r}$		"		$C_2$
	$\mid D_2 \mid$	$A_3$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_{l}$		66		$C_{1}$
			$egin{array}{ccc} C^{j}_{kj} & C^{k}_{jk} & \ C^{j}_{jk} & C^{k}_{jk} & C^{j}_{kj} \ C^{j}_{jk} & C^{k}_{jk} & C^{j}_{kj} \end{array}$					
$R_{ro}$	$\mid D_{11} \mid$	$A_6$	$C^j_{jk}  C^k_{jk}$		$C^k_r$	$C_{kj}^k$		
58 <sub>32</sub>	$D_5$	$A_8$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_{r}$		"		$C_2$
	$D_1$	$A_7$	$C^j_{jk} \ C^k_{jk} \ C^j_{kj}$	$C_{r}$		"		$C_1$
			$C^{j}_{jk} \;\; C^{j}_{jk} \;\; C^{j}_{kj} \ C^{j}_{jk} \;\; C^{j}_{jk} \;\; C^{j}_{jk} \;\; C^{j}_{kj}$					
$R_{-}$	$\mid D_{11} \mid$	$A_6$	$C^j_{jk}  C^k_{jk}$		$C_r^{k}$	$C_{kj}^k$		
± 58 <sub>83</sub>	$\mid D_{\mathtt{5}} \mid$	$A_8$	$C^j_{jk}  C^k_{jk}  C^j_{kj}$	$C_{r}$				$C_2$
		$C_{1}$						
			$C_{kj}^{j}$					
$R_{ m 58_{34}}$	$\mid D_{11} \mid$	$A_6$	$C^{j}_{jk} \ C^{k}_{jk}$ $C_{2}$ $C^{j}_{jk} \ C^{k}_{jk} \ C^{j}_{jk}$		$C^k_r$	$C_{kj}^{k}$		
	$D_2$	$A_3$	$C^j_{jk}$ $C^k_{jk}$ $C^j_{kj}$	$C_{l}$		"		$C_1$

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)	(3)	Proved by (3)	(6)
Nota- tion.	From Table I.	Assumptions.	(4) (5)	Proved by (3) and (4).
		$C^{i}_{kj}$		
$R_{\scriptscriptstyle 58_{35}}$	$D_{11}$	$egin{array}{cccc} A_6 & C^j_{jk} & C^k_{jk} & & & & & & & & & & & & & & & & & & &$	$C_{kj}^{k}$	
	$D_1$	$A_{r} = C^{i}, C^{k}, C^{i}, C$	16	$C_{1}$
	$D_{10}$		$C_l^k$ $C_{jk}^k$	
$R_{ m 66_1}$	$D_{9}$	$A_5$ $C_{kj}^i C_{kj}^k C_r^i$	$C^{j}_{jk}$	
	$D_6$	$egin{array}{cccc} A_4 & & & C_{kj}^j & C_{kj}^k & & C_l \ & & & & & & & & & & & & & & & & & & $	66 66	$C_2$
	$egin{array}{c} D_{10} \ D_{9} \ D_{5} \end{array}$	$oldsymbol{A_6} \qquad \qquad C_{kj}^j \ C_{kj}^k \qquad \qquad C_{kj}^j \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \ \$	$C_{l}^{k}$	·
70	$D_9$	$oldsymbol{A_5} \qquad \qquad C_{kj}^j \; C_{kj}^k \qquad C_r^j$		
$R_{70_1}$	$D_{\mathfrak{b}}$	$A_8 \qquad \qquad C^{g}_{kj} \ C^k_{kj} \ C_r$	66 66.	$C_2$
	• • • •	$C_1$		
	$egin{array}{c} D_{10} \ D_{9} \ \dots \end{array}$	$A_6$ $C_{kj}^j C_{kj}^k$	$C_l^k$ $C_{jk}^k$	
$R_{700}$	$\mid D_{\mathfrak{g}} \mid$	$A_5$ $C^j_{kj} \ C^k_{kj}$ $C^j_r$	$C^{j}_{jk}$	
	$D_2$	$A_3$ $C^j_{bi}$ $C^k_{bi}$ $C_1$	ic cc	$C_1$
	$D_{10}$ $D_{9}$ $\dots$	$m{A_6}$ $m{C_{kj}^i}$ $m{C_{kj}^k}$	$C_{jk}^{k}$	
$R_{70_3}$	$D_{9}$	$A_5$ $C_{kj}^j \ C_{kj}^k \qquad C_r^j$	$C^{j}_{jk}$	
ů	$D_1$	$C_2$ $C_{kj}^j \ C_{kj}^k \ C_r$	"	$C_{1}$

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)			(3)			Proved	b <b>y</b> (3)	(6)
Nota- tion.	From Table I.			Assumptions		(4)	(5)	Proved by (3) and (4).	
	$D_{10}$	$A_6$		$C^{j}_{kj} \; C^{k}_{kj}$	***************************************	$C_l^k$	$C^k_{jk}$		
$R_{ra}$			$C^j_{jk}$						
704	$\mid D_{8} \mid$	$A_4$	$C^j_{jk}$		$*C_l^j$		4.6		$C_2$
	$\mid D_2 \mid$	$A_3$	$C^j_{jk}$	$C^j_{kj} \; C^k_{kj}$	$C_{\iota}$		"		$C_1$
	$D_{10}$	$A_{\mathfrak{6}}$		$C_{kj}^{j} \ C_{kj}^{k}$		$C_{\it i}^{\it k}$	$C^k_{jk}$		
D			$C^j_{jk}$						
$oldsymbol{n}_{70_5}$	$D_8$	$A_4$	$C^j_{jk}$		$C_{\it l}^{\it j}$		"		$C_2$
	$D_1$	$A_7$	$C^j_{jk}$	$C_{kj}^j  C_{kj}^k  C_{kj}^k$	y T		"		$C_1$
	$D_{10}$	$A_6$		$C^j_{kj}  C^k_{kj}$		$C_{\it l}^{\it k}$	$C^k_{jk}$		
70			$C^j_{jk}$						
$R_{70_6}$	$D_7$	$A_8$		$C^j_{kj}  C^k_{kj}$	$C^{j}_{m{r}}$			$C_2$	·
	$D_2$	$A_3$	$C^j_{ik}$	$C^j_{kj}  C^k_{kj}$	$C_{l}$		"		$C_1$
	$D_{10}$	$A_6$		$C^j_{kj}  C^k_{kj}$		$C_i^k$	$C^k_{jk}$		
70			$C^j_{jk}$					$C_2$	
$R_{70_7}$	$D_{7}$	$A_8$		$C^{j}_{kj}$ $C^{k}_{kj}$	$^*C^j_r$			2	B
	$D_1$	$A_{7}$	$C^{j}_{jk}$	$C^{j}_{kj}$ $C^{k}_{kj}$ $C$	y r		"		$C_1$
	$ D_{10} $	$A_6$		$C^{i}_{kj} \ C^{k}_{kj}$		$C_i^k$	$C^k_{jk}$		
70			$C^j_{jk}$						
$R_{70_8}$	$D_6$	$A_4$	$C^j_{jk}$	$C^j_{kj}  C^k_{kj}$	$C_{l}$		"		$C_2$
	$ D_4 $	$A_3$		$C^{j}_{kj}$ $C^{k}_{kj}$		$C_l^k$		$C_1$	

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)			(3)			Proved	by (3)	(6)
Nota- tion.	From Table I.			Assumptions	(4)	(5)	Proved by (3) and (4).		
	$D_{10}$	$A_6$		$C^{j}_{kj}$ $C^{k}_{kj}$		$C_l^k$	$C^k_{jk}$		
D			$C^{j}_{jk}$						
$n_{70_9}$	$D_6$	$A_4$	$C^{j}_{jk}$	$C^j_{kj}  C^k_{kj}$	$C_{l}$		"		$C_2$
	$D_3$	$A_7$	$C^{j}_{jk}$			$C_r^k$	"		$C_1$
	$D_{10}$	$A_6$		$C^j_{kj} \ C^k_{kj}$		$C_l^k$	$C^k_{jk}$		
n			$C^j_{jk}$						
$R_{70_{10}}$	$D_6$	$A_4$	$C^j_{jk}$	$C^{j}_{kj}$ $C^{k}_{kj}$	$C_{\imath}$		"		$C_2$
	$D_2$	$A_3$	$C^j_{jk}$	$C^j_{kj}  C^k_{kj}$	$C_l$		"		$C_1$
	$D_{10}$	$A_6$		$C^{i}_{kj} \ C^{k}_{kj}$		$C_l^k$	$C^k_{jk}$		
			$C^{j}_{jk}$						
$R_{70_{11}}$	$D_6$	$A_4$	$C^j_{jk}$	$C^{j}_{kj}  C^{k}_{kj}$	$C_{l}$		"		$C_2$
	$D_1$	$A_7$	$C^j_{jk}$	$C^j_{kj}$ $C^k_{kj}$ $C$	r		"		$C_1$
	$D_{10}$	$A_6$		$C^{j}_{kj}  C^{k}_{kj}$		$C_{i}^{k}$	$C^k_{jk}$		
_			$C^j_{jm{k}}$						
$R_{70_{12}}$	$D_6$	$A_4$	$C^j_{jk}$	$C^j_{kj} \ C^k_{kj}$	$C_l$		"		$C_2$
		$C_1$							
	$D_{10}$	$A_6$		$C^j_{kj}  C^k_{kj}$ $C^k_{kj}  C^k_{kj}$		$C_{l}^{k}$	$C^k_{jk}$		
<b>T</b> D			$C^j_{jk}$						
$R_{70_{13}}$	$D_5$	$A_8$	$C^{j}_{jk}$	$C^j_{kj}  C^k_{kj}  C$	, <b>,</b>		4.6		$C_2$
	$D_4$	$A_3$		$C^j_{kj}  C^k_{kj}$		$C_l^k$		$C_1$	

TABLE IV<sub>2</sub>—Continued.

(1)	(2)			(3)		Proved	<b>by</b> (3)	(6)
Nota- tion.	From Table I.			Assumptions.		(4)	(5)	Proved by (3) and (4).
	$D_{10}$	$A_6$		$C^{j}_{kj} \ C^{k}_{kj}$	$C_{\iota}^{\prime k}$	$C^k_{jk}$		
$R_{70}$			$C^{j}_{m{j}m{k}}$	$C^j_{kj}  C^k_{kj}  C_r$				
1014	$D_{5}$	$A_8$	$C^{g}_{jk}$	$C^{j}_{kj}  C^{k}_{kj}  C_{r}$		44		$C_2$
	$ig  D_3$	$A_7$	$C^{j}_{jk}$		$*C_r^k$	"		$C_1$
	$D_{10}$	$A_6$		$C_{kj}^{m j}$ $C_{kj}^k$	$C_l^k$	$C^k_{jk}$		
D			$C^j_{jk}$					
$R_{70_{15}}$	$D_5$	$A_8$	$C^{j}_{jm{k}}$	$C^j_{kj} \ C^k_{kj} \ C_{r}$		"		$C_2$
	$D_2$	$A_3$	$C^j_{jk}$	$C^j_{kj}  C^k_{kj} $		"		$C_1$
	$D_{10}$	$A_6$		$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C_{\it i}^{\it k}$	$C^k_{jk}$		
D			$C^j_{jk}$					
<b>11</b> <sub>7016</sub>	$D_5$	$A_8$	$C^j_{jk}$	$C^j_{kj} \ C^k_{kj} \ C_r$		"		$C_2$
	$D_1$	$A_7$	$C^j_{jk}$	$C^{j}_{kj}  C^{k}_{kj}  C_{r}$		"		$C_1$
	$D_{10}$	$A_6$		$egin{array}{ll} C_{kj}^i & C_{kj}^k & C_r \ & C_{kj}^i & C_k^k & C_r \end{array}$	$C_l^k$	$C^{\mathtt{k}}_{jk}$		
D			$C^j_{jk}$				1	
$M_{70_{17}}$	$D_5$	$A_{8}$	$C^j_{jk}$	$C^j_{kj}  C^k_{kj}  C_r$		"		$C_2$
	• • • •	$C_1$						
	$D_{10}$	$A_6$		$egin{array}{ll} C_{kj}^i & C_{kj}^k & & & & & & & & & & & & & & & & & & &$	$C_l^k$	$C^k_{jk}$		
m			$C^j_{\jmath k}$					
$K_{70_{18}}$			$C_2$					
	$\mid D_2 \mid$	$A_3$	$C^j_{jk}$	$C^j_{kj}  C^k_{kj}  C_l$		"		$C_1$

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)		(3)	Proved l	оу (3)	(6)
Nota- tion.	From Table I.		Assumptions.	(4)	(5)	Proved by (3) and (4).
_		$A_6$	$C^{i}_{kj}$ $C^{k}_{kj}$ $C^{k}_{l}$ $C^{i}_{l}$	$C^k_{jk}$		
$R_{70_{19}}$	$D_1$	$A_7$	$C^{i}_{jk} \ C_2 \ C^{i}_{jk}  C^{i}_{kj} \ C^{k}_{kj} \ C_{r}$	"		$C_1$
			$egin{array}{cccc} C^k_{jk} & & & & & & & & & & & & & & & & & & &$			
$R_{ m 70_{20}}$	$egin{array}{c} D_9 \ D_8 \end{array}$	$A_5$	$C_{kj}^{c}$ $C_{kj}^{c}$ $C_{r}^{g}$	$egin{array}{c} C^j_{jk} \ & & & & \end{array}$		
	$ T_{\mathbf{Q}} $	$A_3$	$C^k_{jk}$ $C^i_{kj}$ $C^k_{kj}$ $C_{m l}$	"		$egin{array}{ccc} C_2 \ C_1 \end{array}$
	$egin{array}{c} D_2 \ \dots \ D_9 \ D_8 \ \end{array}$		$C^k_{jk}$			
$R_{70\sigma}$	$ig  D_9$	$A_5$	$C^j_{kj} \; C^k_{kj} \qquad C^j_r$	$C^j_{jk}$		
	$ D_1 $	$A_7$	$egin{array}{ll} C^k_{jk} & C^i_l \ C^k_{jk} & C^i_{kj} & C_r \end{array}$	"		$egin{array}{ccc} C_{f 2} & & & & & & & & & & & & & & & & & & $
			$C^k_{jk}$			
$\boldsymbol{R}$	$egin{array}{c} D_{9} \ D_{7} \end{array}$	$A_5$	$C^{j}_{kj} \; C^{k}_{kj} \qquad  C^{j}_{r}$	$C^{j}_{jk}$		
±070 <sub>22</sub>			$C_{kj}^{i}\;C_{kj}^{k}\qquad C_{r}^{j}$		$C_2$	
	$D_2$	$A_3$	$C^k_{jk} \; C^j_{kj} \; C^k_{kj} \; \; \; C_{m{i}}$	"		$C_1$
			$C^k_{ik}$			
<b>R</b>	$egin{array}{c} D_{9} \ D_{7} \end{array}$	$A_5$	$C^{ij}_{kj}  C^{k}_{kj} $	$C^{j}_{jk}$		
±070 <sub>23</sub>		$A_8$	$C_{kj}^{i} \; C_{kj}^{k} \qquad C_{r}^{j}$		$C_2$	
	$D_1$	$A_7$	$C^k_{jk} \ C^i_{kj} \ C^k_{kj} \ C_r$	"		$C_1$

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)	(3)	Proved b	y (3)	(6)
Nota- tion.	From Table I.	Assumptions.	(4)	(5)	Proved by (3) and (4).
	$D_9$ $D_6$ $D_4$		$C^{j}_{jk}$	$C_1$	$C_2$
	$egin{array}{c} D_9 \ D_6 \ D_3 \end{array}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C^j_{jk}$		$egin{array}{ccc} C_{2} & & & & & & & & & & & & & & & & & & &$
$R_{ m 70_{26}}$	$D_{9}$ $D_{6}$ $D_{2}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C^{j}_{jk}$		$egin{array}{ccc} C_{f 2} & & & & & & \\ C_{f 1} & & & & & & & \end{array}$
$R_{ au_{0_{27}}}$	$egin{array}{c} D_9 \ D_6 \ D_1 \ \end{array}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C^j_{jk}$		$egin{array}{ccc} C_{f 2} & & & & & & & & & & & & & & & & & & $
$R_{70_{23}}$	$egin{array}{c} \dots \ D_9 \ D_6 \ \dots \end{array}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C^{j}_{jk}$ "		$C_2$
$R_{ m 70_{20}}$	$egin{array}{c} D_9 \ D_5 \ D_4 \ \end{array}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C^j_{jk}$	$C_1$	$C_2$

TABLE IV<sub>2</sub>.—Continued.

(1)	(2)	(3)			Proved	by (8)	(6	3)
Nota- tion	From Table I.	Assumptions			(4)	(5)	Prov by (3) an	v
$R_{70_{so}}$	$D_9$ $D_5$ $D_3$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C_r^j$ .	$^*C_r^k$	Cjk		$oxed{C_1}$	$C_2$
$R_{70_{31}}$	$egin{array}{c} D_9 \ D_5 \ D_2 \end{array}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C_r^j$ $C_l$		$C^{i}_{jk}$		$C_{\mathbf{i}}$	$C_{2}$
$R_{70_{32}}$	$egin{array}{c} D_9 \ D_5 \ D_1 \end{array}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C^j_{ au}$		$C^{j}_{jk}$		$oxed{C_1}$	$C_2$
$R_{70_{ss}}$	$egin{array}{c} \dots \ D_9 \ D_5 \ \dots \end{array}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C_r^j$		$C^i_{jk}$			${}^{\prime}C_{2}$
$R_{70_{ m s_4}}$	$egin{array}{c} \dots \ D_9 \ \dots \ D_2 \end{array}$	$C_{2}$	$C_{m{r}}^{j}$ $C_{l}$		$C^i_{jk}$		$C_1$	
$R_{70_{35}}$	$D_{9}$ $D_{1}$	$egin{array}{cccccccccccccccccccccccccccccccccccc$	$C_r^j$		$C^{i}_{jk}$		$oxed{C_1}$	

#### $\S 3.$ —Independence Proofs.

The sets of conditions in  $Tables \ III_2$ . and  $IV_2$ . yield reducibility. It can readily be seen that there are not enough assumptions in any one of these sets to prove the associativity A of the system E. In each case this can be accomplished by adding the assumptions  $A_1$  and  $A_2$ . By the use of  $Table \ V$ , the mutual independence of the conditions composing each of the definitions of  $Table \ III_2$ , and those of all but twelve of the definitions of  $Table \ IV_2$ , can be established. The independencies of  $Table \ V$ , are based upon the following multiplication tables, for which the borders have been omitted.

	I.		]	II.		I	II.	
$e_2$	0	0	$e_1 + e_2$	0	)	$e_1$	0	$e_1$
0	$e_1$	0	0	$e_{i}$	2	0	$e_2$	0
0	0	$e_3$				0	0	$e_3$
$E_j$ =	$=e_1, e_2$	$E_k = e_3$ .	$E_j = \epsilon$	$e_1; E_k$	$=e_2$ .	$E=e_1$	$, \dot{e_2}; E_2$	$e_k = e_3$ .
	I	T.		v.		V	T.	
$e_1$	0	0	$e_1$	$e_{\scriptscriptstyle 1}$	l	0	0	$e_1$
$e_1$	$e_2$		$e_1$	$e_2$	}	$e_1$	$e_{2}$	0
0	0	$e_3$				0	0	$e_3$
$E_j$ =	$=e_1; E$	$G_k = e_2, e_3.$	$E=e_{\underline{c}}$	$_{_{1}};E_{_{k}}$ :	$=e_2$ .	$E_j = e_1$ , $e$	$E_2; E_k =$	$=e_3.$
	V.	II.	,	VIII.		I	X.	
$e_1$	0	$e_3$	$e_1$	0	0	0	0	0
0	$e_2$	0	$e_2$	0	0	$e_{1}$	$e_2$	0
0	$e_{3}$	0	0	0	$e_{3}$	0	0	$e_3$

 $E_j = e_1$ ;  $E_k = e_2$ ,  $e_3$ .  $E_j = e_1$ ,  $e_2$ ;  $E_k = e_3$ .  $E_j = e_1$ ,  $e_2$ ;  $E_k = e_3$ .

TABLE V.

	$A_1$	$A_2$	$A_3$	$A_4$	$A_5$	$A_6$	$A_7$	$A_8$	$C_1$	$C_2$	$C^{j}_{jk}$	$C^k_{jk}$	$\left C_{kj}^{j} ight $	$C_{kj}^k$	$C_l$	$C_r$	$\Big C_l^{u}$	$C_l^k$	$C_r^j$	$C_r^k$	Proof.
1	i	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	I.
2	*	i	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	Interchange $j, k$ in (1).
3	*	*	$i_1$	*	*	*	$i_2$	*	$i_3$	*	*	*	*	*	*	*	*	*	*	*	п.
4	*	*	*	$i_1$	*	*	*	$i_2$	*	$i_3$	*	*	*	*	*	*	*	*	*	*	Interchange $j$ , $k$ in (3).
5	*	*	*	*	$i_1$	*	*	*	*	*	$i_2$	*	*	*	*	*	*	*	*	*	щ.
6	*	*	*	*	$i_1$	*	*	*	*	*	*	*	$i_2$	*	*	*	*	*	*	*	IV.
7	*	*	*	*	*	$ i_1 $	*	*	*	*	*	$i_2$	*	*	*	*	*	*	*	*	Interchange $j$ , $k$ in (6).
8	*	*	*	*	*	$i_1$	*	*	*	*	*	*	*	$i_2$	*	*	*	*	*	*	Interchange $j, k$ in (5).
9	*	*	*	*	*	*	*	*	*	*	$i_1$	*	$i_2$	*	*	*	*	*	*	*	v.
10	*	*	*	*	*	*	*	*	*	*	$i_1$	*	*	*	*	*	*	*	$i_2$	*	VI.
11	*	*	*	*	*	*	*	*	*	*	*	$i_1$	*	$i_2$	*	*	*	*	*	*	Interchange $j, k$ in (9).
12	*	*	*	*	*	*	*	*	*	*	*	$i_1$	*	*	*	*	*	$i_2$	*	*	VII.
13	*	*	*	*	*	*	*	*	*	*	*	*	$i_1$	*	*	*	$i_2$	*	*	*	Interchange $j, k$ in (12).
14	*	*	*	*	*	*	*	*	*	*	*	*	*	$oldsymbol{i_1}$	*	*	*	*	*	$i_2$	Interchange $j$ , $k$ in (10).
15	*	*	*	*	*	*	*	*	*	*	*	*	*	*	$i_1$	*	$i_2$	*	*	*	VIII.
16	*	*	*	*	*	*	*	*	*	*	*	*	*	*	$i_1$	*	*	$i_2$	*	*	Interchange $j, k$ in (15).
17	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	$i_1$	*	*	$i_2$	*	IX.
18	*	*	*	*	*	*	*	*	*	*	*	*	*	*	*	$ i_1 $	*	*	*	$i_2$	Interchange $j, k$ in (17).

#### $\S 4.$ —Dependence Proofs.

The twelve sets of conditions of  $Table\ IV_2$ , that require individual consideration are  $R_{58_7}$ ,  $R_{58_8}$ ,  $R_{58_{14}}$ ,  $R_{58_{20}}$ ,  $R_{58_{22}}$ ,  $R_{58_{24}}$ ,  $R_{70_4}$ ,  $R_{70_7}$ ,  $R_{70_{14}}$ ,  $R_{70_{20}}$ ,  $R_{70_{24}}$ , and  $R_{70_{20}}$ .

1. That  $C_r^j$  is a consequence of the other conditions of  $R_{58_7}$  can be shown in the following manner. The conditions from which  $C_{kj}^j$  is derived in  $D_{12}$  are independent. By  $D_1$  and  $D_5$  it is seen that  $C_1$  and  $C_2$  are consequences of  $A_7$ ,  $A_8$ ,  $C_{jk}^j$ ,  $C_{kj}^k$ ,  $C_{kj}^k$ ,  $C_k^j$ ,  $C_k^k$ ,  $C_r$  and all these conditions are mutually independent. According to the condition  $C_r$ , there exist a J and a K such that  $(J_1 + K_1)(J + K) = 0$ ,

only if  $J_1 = 0 = K_1$ . Multiplying out, we have in view of  $C_{jk}^j$ ,  $C_{jk}^k$ ,  $C_{kj}^j$ ,  $C_{kj}^k$  that

$$J_1J + K_1K = 0$$
, only if  $J_1 = 0 = K_1$   
 $J_3 + K_3 = 0$ , by  $C_1$ ,  $C_2$ .

or

By the linear independence of the units, it follows that

$$J_3 = J_1 J = 0$$
  
 $K_3 = K_1 K = 0$  only if  $\begin{cases} J_1 = 0 \\ K_1 = 0 \end{cases}$ .

The former condition is  $C_{\bullet}^{j}$ .\*

- 2. That  $C_l^k$  is a consequence of the other conditions of  $R_{58}$  can be shown by the successive use of  $D_{12}$ ,  $D_6$ ,  $D_2$ , and  $C_l$ .
- 3. That  $C_r^k$  is a consequence of the other conditions of  $R_{58_{14}}$  can be shown by the successive use of  $D_{12}$ ,  $D_5$ ,  $D_1$ , and  $C_r$ .
- 4. That  $C_i$  is a consequence of the other conditions of  $R_{58_{20}}$  can be shown by the successive use of  $D_{11}$ ,  $D_6$ ,  $D_2$ , and  $C_i$ .
- 5. That  $C_r^j$  is a consequence of the other conditions of  $R_{58_{23}}$  can be shown by the successive use of  $D_{11}$ ,  $D_5$ ,  $D_1$ , and  $C_r$ .
- 6. That  $C_l^k$  is a consequence of the other conditions of  $R_{58_{24}}$  can be shown by the successive use of  $D_{11}$ ,  $D_6$ ,  $D_2$ , and  $C_l$ .
- 7. That  $C_l^i$  is a consequence of the other conditions of  $R_{70_4}$  can be shown by the successive use of  $D_{10}$ ,  $D_6$ ,  $D_2$ , and  $C_l$ .
- 8. That  $C_r^j$  is a consequence of the other conditions of  $R_{70_7}$  can be shown by the successive use of  $D_{10}$ ,  $D_5$ ,  $D_1$ , and  $C_r$ .
- 9. That  $C_r^k$  is a consequence of the other conditions of  $R_{70_{14}}$  can be shown by the successive use of  $D_{10}$ ,  $D_5$ ,  $D_1$ , and  $C_r$ .

$$J_1J + K_1K = 0$$
, only if  $J_1 = 0 = K_1$ , and since  $J_1J = J_3$  and  $K_1K = K_3$ , then  $J'(J_1J + K_1K) = 0$ , only if  $J_1 = 0 = K_1$  and therefore  $J'(J_1J + K_1K) = 0$ , only if  $J_1J = 0 = K_1K$ ,

which gives the required condition.

The reason that this argument is not valid is that the product  $J_3$  (  $\equiv J_1J$ ), although different from zero, may be a right-hand divisor of zero.

By the method used in the body of this paper all the dependence proofs in the previous paper can easily be modified so as to be correct.

<sup>\*</sup>In §5 of the previous paper the corresponding dependence theorems are correctly given. Some of the proofs, however, contain a weakness. Thus, in the above proof, it is not correct to continue the argument: Multiplying on the left by  $\mathcal{F}$ , since

- 10. That  $C_i^i$  is a consequence of the other conditions of  $R_{70_{20}}$  can be shown by the successive use of  $D_9$ ,  $D_6$ ,  $D_2$ , and  $C_i$ .
- 11. That  $C_l^k$  is a consequence of the other conditions of  $R_{70_{24}}$  can be shown by the successive use of  $D_9$ ,  $D_6$ ,  $D_2$ , and  $C_l$ .
- 12. That  $C_r^k$  is a consequence of the other conditions of  $R_{70_{20}}$  can be shown by the successive use of  $D_9$ ,  $D_5$ ,  $D_1$ , and  $C_r$ .

UNIVERSITY OF COLORADO, September, 1905.